Asymptotics of discrete orthogonal polynomials and the continuum limit of the Toda lattice

2001 J. Phys. A: Math. Gen. 34 10627

(http://iopscience.iop.org/0305-4470/34/48/326)

INSTITUTE OF PHYSICS PUBLISHING JOURNAL OF PHYSICS A: MATHEMATICAL AND GENERAL

J. Phys. A: Math. Gen. **34** (2001) 10627–10637 PII: S0305-4470(01)23692-0

Asymptotics of discrete orthogonal polynomials and the continuum limit of the Toda lattice

A I Aptekarev[1] and W Van Assche[2]

[1] Keldysh Institute of Applied Mathematics, Russian Academy of Sciences, Miusskaya Square 4, Moscow 125047, Russia
[2] Department of Mathematics, Katholieke Universiteit Leuven, Celestijnenlaan 200 B, B-3001 Leuven, Belgium

E-mail: walter@wis.kuleuven.ac.be

Received 2 April 2001, in final form 10 July 2001
Published 23 November 2001
Online at stacks.iop.org/JPhysA/34/10627

Abstract
In a recent monograph, Deift and McLaughlin considered the continuum limit of the Toda lattice and solved the problem using perturbation techniques, such as the Wentzel–Kramers–Brillouin (WKB) method. We present an alternative approach based on the analytic theory of discrete orthogonal polynomials. The method is based on some extremal problems in the theory of logarithmic potentials and relies on some results for the asymptotic theory of orthogonal polynomials with a discrete orthogonality measure.

PACS number: 05.50.+q

Mathematics Subject Classification: 42C05, 37K10

1. Introduction

In a recent monograph, Deift and McLaughlin [4] considered the continuum limit of the Toda lattice and solved the problem using perturbation techniques, such as the Wentzel–Kramers–Brillouin (WKB) method. We present an alternative approach based on the analytic theory of discrete orthogonal polynomials [5, 6, 8, 9, 12].

The *finite Toda lattice* can be analysed using the finite chain of differential equations

$$\begin{aligned} \frac{\mathrm{d}a_{k,N}}{\mathrm{d}t} &= (b_{k,N}^2 - b_{k-1,N}^2) \qquad && k = 1, 2, \ldots, N \\ \frac{\mathrm{d}b_{k,N}}{\mathrm{d}t} &= \frac{b_{k,N}}{2}(a_{k+1,N} - a_{k,N}) \qquad && k = 1, 2, \ldots, N-1 \end{aligned} \tag{1}$$

with $b_{0,N} = b_{N,N} = 0$ and with initial data

$$a_{k,N}(0) \quad b_{k,N}(0) \qquad 1 \leqslant k \leqslant N. \tag{2}$$

The *Cauchy problem* consists of finding $a_{k,N}(t), b_{k,N}(t)$ for $t > 0$ satisfying the differential equations (1) with initial data (2).

The continuum limit is when the limits

$$\lim_{N\to\infty,k/N\to x} a_{k,N}(0) = a(x) \qquad \lim_{N\to\infty,k/N\to x} b_{k,N}(0) = b(x) \tag{3}$$

exist uniformly for $0 \leqslant x \leqslant 1$, and in this case we want to obtain the limits

$$\lim_{N\to\infty,k/N\to x} a_{k,N}(Nt) = a(x,t) \qquad \lim_{N\to\infty,k/N\to x} b_{k,N}(Nt) = b(x,t) \tag{4}$$

for $0 < x < 1$ and $t > 0$. Under appropriate conditions these limits $a(x, t), b(x, t)$ satisfy the system of partial differential equations

$$\begin{aligned} \frac{\partial a}{\partial t} &= 2b\,\frac{\partial b}{\partial x} \\ \frac{\partial b}{\partial t} &= \frac{b}{2}\frac{\partial a}{\partial x} \end{aligned} \tag{5}$$

with initial conditions $a(x, 0) = a(x)$, $b(x, 0) = b(x)$, and boundary condition $b(0, t) = 0 = b(1, t)$. If we put

$$\alpha(x,t) = a(x,t) - 2b(x,t) \qquad \beta(x,t) = a(x,t) + 2b(x,t) \tag{6}$$

then these functions satisfy

$$\begin{aligned} \frac{\partial \alpha}{\partial t} &= -\frac{\beta-\alpha}{4}\,\frac{\partial \alpha}{\partial x} \\ \frac{\partial \beta}{\partial t} &= \frac{\beta-\alpha}{4}\frac{\partial \beta}{\partial x}. \end{aligned} \tag{7}$$

This system is called *the continuum limit of the Toda lattice*. It gives a nice example of a model hyperbolic PDE, which for $\alpha \equiv 0$ reduces to the well-known inviscid Burgers equation.

The finite discrete system (1) was solved by Moser [10, 11] and the solution can be presented in terms of discrete orthogonal polynomials. Define the Jacobi matrix

$$L_N(t) = \begin{pmatrix} a_{1,N}(t) & b_{1,N}(t) & & & & \\ b_{1,N}(t) & a_{2,N}(t) & b_{2,N}(t) & & & \\ & b_{2,N}(t) & a_{3,N}(t) & b_{3,N}(t) & & \\ & & \ddots & \ddots & \ddots & \\ & & & b_{N-2,N}(t) & a_{N-1,N}(t) & b_{N-1,N}(t) \\ & & & & b_{N-1,N}(t) & a_{N,N}(t) \end{pmatrix} \tag{8}$$

with eigenvalues $\lambda_{j,N}(t)$ $(j = 1, 2, \ldots, N)$. This Jacobi matrix gives a finite system of orthonormal polynomials through the recurrence relation

$$\lambda p_{k,N}(\lambda,t) = b_{k+1,N}(t)p_{k+1,N}(\lambda,t) + a_{k+1,N}(t)p_{k,N}(\lambda,t) + b_{k,N}(t)p_{k-1,N}(\lambda,t) \tag{9}$$

for $0 \leqslant k \leqslant N-2$, and

$$p_{N,N}(\lambda,t) = (\lambda - a_{N,N}(t))p_{N-1,N}(\lambda,t) - b_{N-1,N}(t)p_{N-2,N}(\lambda,t)$$

with initial conditions $p_{0,N} \equiv 1$ and $p_{-1,N} \equiv 0$. These polynomials (in the variable λ) are orthonormal with respect to a discrete measure μ_N supported on the eigenvalues $\{\lambda_{j,N}(t) : j = 1, 2, \ldots, N\}$, which are precisely the zeros of $p_{N,N}(\lambda, t)$:

$$\sum_{j=1}^{N} p_{k,N}(\lambda_{j,N})p_{\ell,N}(\lambda_{j,N})w_{j,N} = \delta_{k,\ell} \qquad 0 \leqslant k, \ell \leqslant N-1 \tag{10}$$

where

$$w_{j,N} = \frac{1}{\sum_{k=0}^{N-1} p_{k,N}^2(\lambda_{j,N})}. \tag{11}$$

For convenience we will use the discrete measure

$$\mu_N(\lambda, t) = \sum_{j=1}^{N} w_{j,N}(t)\delta(\lambda - \lambda_{j,N}(t)) \tag{12}$$

so that the orthonormality (10) is written as

$$\int p_{k,N}(\lambda, t) p_{\ell,N}(\lambda, t)\,\mathrm{d}\mu_N(\lambda, t) = \delta_{k,\ell} \qquad 0 \leqslant k, \ell \leqslant N-1. \tag{13}$$

Theorem 1 (Moser). *If $a_{k,N} \in \mathbb{R}$ $(1 \leqslant k \leqslant N)$ and $b_{k,N} > 0$ $(1 \leqslant k \leqslant N-1)$, then*

(i) The system (1) has a global solution for any $t > 0$.
(ii) The solution can be obtained by the following procedure:

- *First find the eigenvalues $\lambda_{j,N}$ $(1 \leqslant j \leqslant N)$ and the weights $w_{j,N}$ $(1 \leqslant j \leqslant N)$ for the Jacobi matrix $L_N(0)$ containing the initial data (the direct spectral problem).*
- *Take the evolution of the spectral data*

$$\frac{\mathrm{d}\lambda_{j,N}}{\mathrm{d}t} = 0$$

$$w_{j,N}(t) = \frac{\mathrm{e}^{\lambda_{j,N}t} w_{j,N}}{\sum_{k=1}^{N} \mathrm{e}^{\lambda_{k,N}t} w_{k,N}}$$

so that

$$\mathrm{d}\mu_N(\lambda, t) = \frac{\mathrm{e}^{\lambda t}\,\mathrm{d}\mu_N(\lambda, 0)}{\int \mathrm{e}^{yt}\,\mathrm{d}\mu_N(y, 0)}. \tag{14}$$

- *Obtain the Jacobi matrix $L_N(t)$ for $\mu_N(\lambda, t)$ (the inverse spectral problem), e.g., by expanding the rational function*

$$\int \frac{\mathrm{d}\mu_N(\lambda, t)}{z - \lambda}$$

into a terminating continued fraction (J-fraction).

Thus the procedure of integrating the Toda equations (1) is given by the following scheme

$$\begin{tikzcd}
\boxed{\text{operator data at } t=0} \arrow[r, "\text{Toda equation (1)}"] \arrow[d, "\text{direct spectral problem}"] & \boxed{\text{operator data at } t} \\
\boxed{\text{spectral data at } t=0} \arrow[r, "\text{evolution of spectral data (14)}"] & \boxed{\text{spectral data at } t} \arrow[u, "\text{inverse spectral problem}"']
\end{tikzcd}$$

2. Operator and spectral data for the continuum limit

For the continuum limit (5) we take the *operator data* as $a(x), b(x) \in C[0, 1]$ (or $\alpha(x), \beta(x) \in C[0, 1]$; see (6)). For the finite discrete problem the operator data are

$$a_{k,N} = a(k/N) \qquad b_{k,N} = b(k/N).$$

It is clear that

$$\lim_{N\to\infty} a_{[xN],N} = a(x) \qquad \lim_{N\to\infty} b_{[xN],N} = b(x) \tag{15}$$

holds uniformly for $x \in [0, 1]$, where $[xN]$ is the greatest integer less than or equal to xN.

The spectral data for the discrete finite system $\lambda_{j,N}$, $w_{j,N}$ $(j = 1, \ldots, N)$ are collected in the discrete measure μ_N. We define the spectral data for the continuum limit as the measure σ and the function $Q \in C[\mathrm{supp}(\sigma)]$ satisfying

$$\sigma(\lambda) = \lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} \delta(\lambda - \lambda_{j,N}) \tag{16}$$

$$\lim_{N\to\infty} \max_{1\leqslant j\leqslant N} \left| \frac{1}{N} \log w_{j,N} - Q(\lambda_{j,N}) \right| = 0. \tag{17}$$

3. Inverse problem for the continuum limit

The inverse spectral problem for the continuum limit is to find the operator data $a(x)$, $b(x)$ if the spectral data (σ, Q) are given. In terms of the discrete orthogonal polynomials the inverse problem is to find the limits (15) of the coefficients of the recurrence relation (9) of the orthonormal polynomials satisfying (13) by means of information about the limit behaviour (16), (17) of the orthogonality measure. This problem can be solved by finding appropriate asymptotics for the orthogonal polynomials. The basic known result in this direction was obtained by Rakhmanov [12] for $Q = 0$ and extended for a general class of functions Q by Dragnev and Saff [5, 6] (see also [8]). It is more convenient to work with monic orthogonal polynomials $P_{k,N}(\lambda) = \lambda^k + \cdots$ which satisfy the recurrence relation

$$\lambda P_{k,N}(\lambda) = P_{k+1,N}(\lambda) + a_{k+1,N} P_{k,N}(\lambda) + b_{k,N}^2 P_{k-1,N}(\lambda) \tag{18}$$

with $P_{0,N} = 1$ and $P_{1,N} = \lambda - a_{1,N}$.

Theorem 2 (Rakhmanov, Dragnev–Saff). *If the orthogonality measures (12) for the families of discrete orthonormal polynomials (10) satisfy the limiting conditions (16) and (17), together with a technical condition on the eigenvalues (a 'separation condition'):*

$$|\lambda_{j+1,N} - \lambda_{j,N}| > \mathcal{O}(1/N) \qquad k = 1, \ldots, N-1$$

then the following asymptotic formula

$$\lim_{N\to\infty} |P_{[xN],N}(z)|^{1/n} = \exp\left(-\frac{1}{x} V_{\tau_x}(z)\right) \tag{19}$$

holds uniformly for z belonging to compact sets of $\overline{\mathbb{C}} \setminus \mathrm{supp}(\tau_x)$ and $x \in [0, 1]$. Here

$$V_\tau(z) = \int \log \frac{1}{|z-\lambda|} \,\mathrm{d}\tau(\lambda)$$

is the logarithmic potential of a measure τ and the family of measures τ_x, $x \in [0, 1]$ solves the extremal problem of minimizing the logarithmic energy with external field Q given by (17) for measures μ of total mass x, constrained by the measure σ given in (16), i.e.,

$$W_Q(\tau_x) = \inf_{\mu(\mathbb{R})=x,\ \mu\leqslant\sigma} W_Q(\mu) \tag{20}$$

with

$$W_Q(\mu) = \iint \log \frac{1}{|s-t|} \,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) - \int Q(t)\,\mathrm{d}\mu(t).$$

The separation condition can be weakened considerably (see [1]) but this is not so relevant in our analysis. The family of extremal measures τ_x, $x \in [0, 1]$ satisfies the following equilibrium condition (see [5,6,8,12]):

Theorem 3 (Rakhmanov, Dragnev–Saff). *The solution of the extremal problem (20) exists and is unique. Furthermore this solution is characterized by the following relations*

$$V_{\tau_x}(\lambda) - \frac{1}{2}Q(\lambda) \begin{cases} \geqslant \gamma_x & \lambda \in \mathrm{supp}(\sigma - \tau_x) \\ = \gamma_x & \lambda \in \mathrm{supp}(\sigma - \tau_x) \cap \mathrm{supp}(\tau_x) := \Sigma_x \\ \leqslant \gamma_x & \lambda \in \mathrm{supp}(\tau_x). \end{cases} \tag{21}$$

This means that, for each $x \in [0, 1]$, the equilibrium conditions (21) uniquely define the extremal constant γ_x and the extremal measure τ_x for the problem (20).

If the set Σ_x for the extremal measure τ_x is an interval:

$$\Sigma_x = [\alpha(x), \beta(x)] \tag{22}$$

then, as we shall show later, the right-hand side of the asymptotic formula (19) contains the solution of the direct problem. If the limits of the recurrence coefficients (15) exist, then they are given by

$$\lim_{N\to\infty} a_{[xN],N} = \frac{\alpha(x) + \beta(x)}{2} \qquad \lim_{N\to\infty} b_{[xN],N} = \frac{\beta(x) - \alpha(x)}{4}. \tag{23}$$

However, to justify the existence of these limits, we need to have a stronger asymptotic formula for the orthonormal polynomials. Actually we need an asymptotic formula for the ratio of two consecutive discrete orthogonal polynomials. Note that the limit of the ratio of two consecutive terms of a sequence does not always exist when the nth root of the sequence exists, but if the limit of the ratio exists, then it is related to the limit of the nth root.

The ratio asymptotics for discrete orthogonal polynomials is still an open problem. The corresponding result for polynomials orthogonal with respect to a continuous measure on an interval was obtained by Rakhmanov [13]. It would be useful to prove an analogous result (under appropriate conditions) for the discrete case.

In this paper we give a conditional solution of the inverse problem in the following sense:

Theorem 4. *If the continuum limit operator data (15) exist, then the limit functions $a(x)$ and $b(x)$ can be obtained from the continuum limit spectral data (16) and (17) by means of (21)–(23).*

Before stating a conditional theorem about the ratio asymptotics of the discrete orthogonal polynomials, we rewrite the formula for the nth-root asymptotics (19) using the following representation of the extremal measure τ_x.

Theorem 5 (see [2,3,8]). *The density of the extremal measure is given by*

$$\tau_x'(\lambda) = \int_0^x \omega_{\Sigma_y}(\lambda)\,\mathrm{d}y \tag{24}$$

where $\omega_{[\alpha,\beta]}$ is the usual density of the equilibrium measure (Chebyshev measure) for the interval $[\alpha, \beta]$

$$\omega_{[\alpha,\beta]}(\lambda) = \frac{1}{\pi}\frac{1}{\sqrt{(\lambda-\alpha)(\beta-\lambda)}} \qquad \lambda \in [\alpha, \beta].$$

The formula (24) for the unconstrained extremal measure was first obtained by Buyarov in [2] (for details, see [3]). It was obtained for the constrained extremal measure under rather strong restrictions (see theorem 9.2 from [8] and [4]). It is still an open problem to prove (24) in a general setting. As a corollary of theorems 3 and 5 we have

$$\lim_{N\to\infty} |P_{[xN],N}(z)|^{1/n} = \exp\left(-\frac{1}{x}\int_0^x V_{\omega_{\Sigma_y}}(z)\,\mathrm{d}y\right) \tag{25}$$

uniformly on compact subsets of $\overline{\mathbb{C}} \setminus \operatorname{supp}(\tau_x)$ and $x \in [0, 1]$.

Now we are ready to state and prove a conditional theorem about ratio asymptotics.

Theorem 6. *If the orthogonality measures (12) for the families of discrete orthogonal polynomials (10) satisfy the conditions of theorem 2, with the result that the asymptotic formula (25) is valid, and if uniformly for z belonging to some compact set K and for $x \in [0, 1]$ the limit of $P_{[xN]-1,N}(z)/P_{[xN],N}(z)$ exists when $N \to \infty$, then this limit is given by*

$$\lim_{N\to\infty} \frac{P_{[xN]-1,N}(z)}{P_{[xN],N}(z)} = \exp\left(U_{\omega_{\Sigma_x}}(z)\right) \tag{26}$$

locally uniformly for $z \in \overline{\mathbb{C}} \setminus \operatorname{supp}(\tau_x)$ and $x \in (0, 1)$, where $U_\omega = V_\omega + \mathrm{i}\tilde{V}_\omega$ is the complex potential of the measure ω.

Proof. We just give an outline of the derivation of (26) from the existence of the limit on the left-hand side of (26) and from (25). We have for $\tilde{n} = [xN]$

$$\frac{1}{\tilde{n}} \log|P_{\tilde{n},N}| = \frac{1}{\tilde{n}} \sum_{n=1}^{\tilde{n}} \log\left|\frac{P_{n,N}}{P_{n-1,N}}\right|.$$

Using the notation $\tilde{x} = \tilde{n}/N$ we have from (25)

$$\frac{1}{\tilde{x}}\int_0^{\tilde{x}} \log\left|\frac{P_{[xN],N}}{P_{[xN]-1,N}}\right| \mathrm{d}x \to -\frac{1}{x}\int_0^x V_{\omega_{\Sigma_x}}\,\mathrm{d}x \qquad \text{as } N \to \infty.$$

Therefore if the limit on the left-hand side of (26) exists, then it is equal to the right-hand side of (26). □

Finally, from the ratio asymptotics (26) we can deduce the formulae for the limits (15) of the coefficients of the recurrence relation (9).

Theorem 7. *If (26) holds, then*

$$\lim_{n\to\infty, n/N\to x\in[0,1]} a_{n,N} = a(x) = \frac{\beta(x)+\alpha(x)}{2}$$
$$\lim_{n\to\infty, n/N\to x\in[0,1]} b_{n,N} = b(x) = \frac{\beta(x)-\alpha(x)}{2}.$$

Proof. The limits above are an immediate corollary of the ratio asymptotics (26) and recurrence relation (18)

$$z - a_{n+1,N} = \frac{P_{n+1,N}(z)}{P_{n,N}(z)} + b_{n,N}^2 \frac{P_{n-1,N}(z)}{P_{n,N}(z)}.$$

Letting $N \to \infty$ here and taking into account that for the equilibrium measure of the interval $\Sigma_x = [\alpha_(x), \beta(x)]$ it is known that

$$\exp\left(-U_{\omega_{\Sigma_x}}(z)\right) = \Phi(z) := \frac{z-a}{2} + \sqrt{\frac{(z-a)^2}{4} - b^2}$$
$$a = \frac{\beta(x)+\alpha(x)}{2} \qquad b = \frac{\beta(x)-\alpha(x)}{4}$$

with the result that $\Phi^2 - (z-a)\Phi + b^2 = 0$, we obtain the desired formulae. □

4. Evolution of the spectral data

To find the evolution of the spectral data for the continuum limit, we substitute in the definitions (16) and (17) the evolution of the discrete spectral data (14) from theorem 1. Thus we have for the constraining measure

$$\sigma(\lambda, t) = \sigma(\lambda, 0) \qquad t > 0 \tag{27}$$

and for the external field

$$Q(\lambda, t) = \lambda t + Q(\lambda, 0) - \max_{\lambda}[\lambda t + Q(\lambda, 0)]. \tag{28}$$

The latter follows since (see (14))

$$\frac{1}{N}\log w_{j,N}(t) = \frac{1}{N}\lambda_{j,N}t + \frac{1}{N}\log w_{j,N} - \frac{1}{N}\log\left(\sum_{k=1}^{N} \mathrm{e}^{\lambda_{k,N}t} w_{k,N}\right).$$

After rescaling $t \to Nt$ this gives

$$\frac{1}{N}\log w_{j,N}(Nt) = \lambda_{j,N}t + Q(\lambda_{j,N}, 0) + \frac{1}{N}\log w_{j,N} - Q(\lambda_{j,N}, 0)$$
$$- \frac{1}{N}\log\sum_{k=1}^{N}\exp\left[N\left(\lambda_{k,N}t + Q(\lambda_{k,N}, 0) + \frac{1}{N}\log w_{k,N} - Q(\lambda_{k,N}, 0)\right)\right].$$

Letting $N \to \infty$ and taking $j(N)$ in such a way that $\lambda_{j(N),N} \to \lambda$, we use the definition of $Q(\lambda)$ (see (17)) and the discrete version of Laplace's asymptotic formula to find

$$\frac{1}{N}\log\sum_{k=1}^{N}\exp[N(\lambda_{k,N}t + Q(\lambda_{k,N}, 0))] \to \max_{\lambda}(\lambda t + Q(\lambda, 0)).$$

Thus, recalling again (17), we arrive at (28) in the continuum timescale.

5. The direct problem for the continuum limit

The direct spectral problem for the continuum limit is to find the spectral data (σ, Q) if the operator data $a(x), b(x)$ (or $\alpha(x) = a(x) - 2b(x)$ and $\beta(x) = a(x) + 2b(x)$) are given. We can solve the direct problem (under a rather strong restriction on the operator data) by combining some known results.

The first step, the determination of the constraining measure σ, follows from a result of Kuijlaars and Van Assche [9]

Theorem 8. *Suppose that $a(x)$ and $b(x) > 0$ are continuous on $(0, 1)$; then*

$$\sigma'(\lambda) = \int_0^1 \omega_{[\alpha(y),\beta(y)]}(\lambda)\,\mathrm{d}y \tag{29}$$

where

$$\omega_{[\alpha,\beta]}(\lambda) = \begin{cases} \dfrac{1}{\pi}\dfrac{1}{\sqrt{(x-\alpha)(\beta-x)}} & \lambda \in (\alpha, \beta) \\ 0 & \lambda \notin (\alpha, \beta). \end{cases}$$

We observe that theorem 8 does not contain any additional restriction on the operator data $a(x), b(x)$. However, for the second step (the determination of the external field Q) we use a known result for some special class of operator data. Following Deift and McLaughlin [4], we assume that the set

$$\{x \in [0, 1] : \lambda \in [\alpha(x), \beta(x)]\}$$

is an interval for each λ, which we denote by $[x_-(\lambda), x_+(\lambda)]$

$$[x_-(\lambda), x_+(\lambda)] = \{x \in [0, 1] : \lambda \in [\alpha(x), \beta(x)]\}. \tag{30}$$

Then the following result holds

Theorem 9 (see [8]). *Suppose that the continuum operator data $a(x)$, $b(x)$ are such that (30) holds. Then*

$$\tau_x'(\lambda) = \int_0^x \omega_{\Sigma_y}(\lambda)\,\mathrm{d}y$$

is a solution of the extremal problem (20), (21) with external field

$$-\tfrac{1}{2}Q(\lambda) = 2V_{\tau_x}(\lambda) - 2\gamma_x \qquad \gamma_x = -\int_0^x \log b(y)\,\mathrm{d}y \qquad \lambda \in [\alpha(x), \beta(x)] \tag{31}$$

and in this case, equation (31) can be rewritten as

$$-\tfrac{1}{2}Q(\lambda) = -2\int_0^{x_-(\lambda)} g_{[\alpha(y),\beta(y)]}(\lambda)\,\mathrm{d}y \tag{32}$$

where $g_{[\alpha,\beta]}(\lambda)$ is the Green function of the interval $[\alpha, \beta]$:

$$g_{[\alpha,\beta]}(\lambda) = \log\left|\frac{\lambda - a}{2b} + \sqrt{\left(\frac{\lambda - a}{2b}\right)^2 - 1}\right|. \tag{33}$$

Thus formulae (29) and (32) give the solution of the direct problem.

6. Conclusions

The asymptotic theory of discrete orthogonal polynomials gives some 'justification' (of course not fully rigorous and with some rather strong restrictions) for the following procedure of integration of the system (7) with initial values $\alpha(x, 0) = \alpha(x)$ and $\beta(x, 0) = \beta(x)$ and boundary values $\beta(0, t) = \alpha(0, t)$ and $\beta(1, t) = \alpha(1, t)$.

- Firstly, from the initial data $\alpha(x, 0) = \alpha(x)$ and $\beta(x, 0) = \beta(x)$, $x \in [0, 1]$, we obtain by means of the formulae (29), (30), (32), (33) the spectral data at time $t = 0$:

$$\sigma'(\lambda) = \int_0^1 \omega_{[\alpha(y),\beta(y)]}(\lambda)\,\mathrm{d}y$$

$$Q(\lambda) = -2\int_0^{x_-(\lambda)} g_{[\alpha(y),\beta(y)]}(\lambda)\,\mathrm{d}y$$

  where

$$\omega_{[\alpha,\beta]}(\lambda) = \begin{cases} \dfrac{1}{\pi}\dfrac{1}{\sqrt{(x-\alpha)(\beta - x)}} & \lambda \in (\alpha, \beta) \\ 0 & \lambda \notin (\alpha, \beta) \end{cases}$$

$$[x_-(\lambda), x_+(\lambda)] = \{x \in [0, 1] : \lambda \in [\alpha(x), \beta(x)]\}$$

$$g_{[\alpha,\beta]}(\lambda) = \log\left|\frac{\lambda - a}{2b} + \sqrt{\left(\frac{\lambda - a}{2b}\right)^2 - 1}\right|.$$

- Next, we apply the evolution on the spectral data by means of formulae (27) and (28)

$$\sigma(\lambda, t) = \sigma(\lambda, 0) \qquad t > 0$$

$$Q(\lambda, t) = \lambda t + Q(\lambda, 0) - \max_\lambda[\lambda t + Q(\lambda, 0)]$$

  up to time $t > 0$.

- Finally, using $\sigma(\lambda, t)$ as the constraint and $Q(\lambda, t)$ as the external field in the extremal problem of the logarithmic potential (20):

$$W_Q(\tau_x) = \inf_{\mu(\mathbb{R})=x,\ \mu\leqslant\sigma} W_Q(\mu)$$

with

$$W_Q(\mu) = \iint \log\frac{1}{|s-t|}\,\mathrm{d}\mu(s)\,\mathrm{d}\mu(t) - \int Q(t)\,\mathrm{d}\mu(t)$$

we obtain the interval Σ_x of the support of equilibrium for the potential of the extremal measure τ_x (see (21))

$$V_{\tau_x}(\lambda) - \tfrac{1}{2}Q(\lambda)\begin{cases} \geqslant\gamma_x & \lambda\in\operatorname{supp}(\sigma-\tau_x) \\ =\gamma_x & \lambda\in\operatorname{supp}(\sigma-\tau_x)\cap\operatorname{supp}(\tau_x) := \Sigma_x \\ \leqslant\gamma_x & \lambda\in\operatorname{supp}(\tau_x) \end{cases}$$

where

$$V_\mu(z) = \int\log\frac{1}{|z-\lambda|}\,\mathrm{d}\mu(\lambda).$$

The extreme points of the interval

$$\Sigma_x(t) = [\alpha(x,t), \beta(x,t)]$$

are the solution of the problem (7) at time $t > 0$.

7. An example: Krawtchouk polynomials

Consider the initial data

$$\alpha(x) = (1-x)p + x(1-p) - 2\sqrt{p(1-p)x(1-x)} \tag{34}$$

$$\beta(x) = (1-x)p + x(1-p) + 2\sqrt{p(1-p)x(1-x)} \tag{35}$$

where $0 < p < 1$. When we plot both functions, we see that they are the lower part (α) and the upper part (β) of an ellipse inscribed in the unit square $[0, 1]^2$. If we take as discrete operator data

$$a_{n,N} = (N-n)p + (n-1)(1-p) \tag{36}$$

$$b_{n,N} = \sqrt{(N-n)kp(1-p)} \tag{37}$$

then clearly

$$\lim_{N\to\infty,\ n/N\to x} a_{n,N}/N = (1-x)p + x(1-p)$$

$$\lim_{N\to\infty,\ n/N\to x} b_{n,N}/N = \sqrt{p(1-p)x(1-x)}$$

which corresponds to the continuum operator data (34), (35). The discrete operator data (36), (37) contain the recurrence coefficients of the orthonormal Krawtchouk polynomials $k_n(\lambda, p, N-1)$, which are orthonormal polynomials with respect to the binomial distribution on $\{0, 1, 2, \ldots, N-1\}$ (see, e.g., [6,9]). Therefore, after scaling, the spectrum is $\lambda_{j,N} = (j-1)/N$ and the weights are

$$w_{j,N} = \binom{N-1}{j-1}p^{j-1}(1-p)^{N-j} \qquad j = 1, \ldots, N.$$

The direct spectral problem for the initial data is therefore easy in this case. We can now find the evolution of the spectral data, which consists simply of replacing the parameter p by

$$p(t) = \frac{p\mathrm{e}^t}{p\mathrm{e}^t + (1-p)}.$$

Observe that $p(0) = p$, $\lim_{t\to\infty} p(t) = 1$, and $\lim_{t\to-\infty} p(t) = 0$. For the discrete system this still gives Krawtchouk polynomials but now with parameter $p(t)$. The inverse spectral problem is therefore also easy and gives the operator data (36), (37), but with parameter $p(t)$. The continuum limit finally gives

$$\alpha(x,t) = (1-x)p(t) + x(1-p(t)) - 2\sqrt{p(t)(1-p(t))x(1-x)}$$
$$\beta(x,t) = (1-x)p(t) + x(1-p(t)) + 2\sqrt{p(t)(1-p(t))x(1-x)}.$$

One can indeed verify that they satisfy the system (7) with initial data (34), (35). The initial ellipse rotates and, as $t \to \infty$, evolves to a straight line through the points (0, 1) and (1, 0). As $t \to -\infty$ the ellipse rotates in the anticlockwise direction and evolves to a straight line through the points (0, 0) and (1, 1). In Maple code, one has

```
> alpha:=x->(1-x)*p+x*(1-p)-2*sqrt(p*(1-p)*x*(1-x));
> beta:=x->(1-x)*p+x*(1-p)+2*sqrt(p*(1-p)*x*(1-x));
> p:=2/5;
> plot([alpha(x),beta(x)],x=0..1,color=[red,blue]);
[ This gives a plot of the initial ellipse, with p=0.4
> pp:=t->p*exp(t)/(p*exp(t)+1-p);
[ Evolution of the parameter p
> alpha:=(x,t)->(1-x)*pp(t)+x*(1-pp(t))-2*sqrt(pp(t)*(1-pp(t))*x*(1-x));
> beta:=(x,t)->(1-x)*pp(t)+x*(1-pp(t))+2*sqrt(pp(t)*(1-pp(t))*x*(1-x));
> with(plots):
> animate([alpha(x,t),beta(x,t)],x=0..1,t=0..8,color=red);
[ Evolution of the ellipses
```

For this example we did not need to solve the constrained equilibrium problem with external field, since everything was quite explicit here. This constrained equilibrium problem for Krawtchouk polynomials was considered in detail by Dragnev and Saff [6] for the spectral data, and by Kuijlaars and Van Assche (see section 4.4 on pp 186–8 of [9]) for the operator data. The corresponding Toda lattice was also considered as an example by Kuijlaars and McLaughlin [7].

Acknowledgment

This work was supported by INTAS 00-272 and was started while the first author was visiting the Katholieke Universiteit Leuven with fellowship F/99/009 of the research counsel of K U Leuven.

References

[1] Beckermann B 2000 On a conjecture of E A Rakhmanov *Constr. Approx.* **16** 427–48
[2] Buyarov V S 1991 Logarithmic asymptotics of polynomials orthogonal on the real axis with nonsymmetric weight *Mat. Zametki* **50** 28–36 (in Russian) (Engl. transl. *Math. Notes* **50** 789–95)
[3] Buyarov V S and Rakhmanov E A 1999 Families of equilibrium measures in an external field on the real axis *Mat. Sb.* **190** 11–22 (in Russian) (Engl. transl. *Sb. Math.* **190** 791–802)
[4] Deift P and McLaughlin K T-R 1998 A continuum limit of the Toda lattice *Mem. Am. Math. Soc.* vol 624 (Providence, RI: American Mathematical Society)

[5] Dragnev P and Saff E B 1997 Constrained energy problems with applications to orthogonal polynomials of a discrete variable *J. Anal. Math.* **72** 223–59

[6] Dragnev P D and Saff E B 1999 A problem in potential theory and zero asymptotics of Krawtchouk polynomials *J. Approx. Theory* **102** 120–40

[7] Kuijlaars A B J and McLaughlin K T-R 2001 Long time behaviour of the continuum limit of the Toda lattice, and the generation of infinitely many gaps from C^∞ initial data *Commun. Math. Phys.* **221** 305–33

[8] Kuijlaars A B J and Rakhmanov E A 1998 Zero distributions for discrete orthogonal polynomials *J. Comput. Appl. Math.* **99** 255–74

[9] Kuijlaars A B J and Van Assche W The asymptotic zero distribution of orthogonal polynomials with varying recurrence coefficients *J. Approx. Theory*

[10] Moser J 1975 Three integrable Hamiltonian systems connected with isospectral deformations *Adv. Math.* **16** 197–220

[11] Moser J 1975 Finitely many mass points on the line under the influence of an exponential potential—an integrable system *Dynamical Systems, Theory and Applications (Rencontres, Battelle Research Institute, Seattle, WA, 1974) (Springer Lecture Notes in Physics vol 38)* (Berlin: Springer) pp 467–97

[12] Rakhmanov E A 1996 Equilibrium measure and the distribution of zeros of the extremal polynomials of a discrete variable *Mat. Sb.* **187** 109–24 (in Russian) (Engl. transl. *Sb. Math.* **187** 1213–28)

[13] Rakhmanov E A 1977 On the asymptotics of the ratio of orthogonal polynomials *Mat. Sb.* **103** 237–52 (in Russian) (Engl. transl. *Math. USSR Sb.* **32** 199–214)